Distribution of Fourier power spectrum of climatic background noise

Zhihua Zhang*, John Moore†‡§
*College of Global Change and Earth System Science, Beijing Normal University Beijing, China
Email: zhangzh@bnu.edu.cn
† State Key Laboratory of Earth Surface Processes and Resource Ecology/College of Global Change and Earth System Science, Beijing Normal University, China
‡Arctic Centre, University of Lapland, Finland
§ Department of Earth Sciences, Uppsala University, Sweden

***Abstract*—In climatic signal processing, significance tests are used to extract statistically significant feature from randomness. For many climatic signals, an appropriate background noise is red noise. In 1998, Torrence and Compo [1] provided an empirical formula on the distribution of Fourier power spectrum of red noise which is the foundation of significance tests on Fourier analysis of climatic signals. In this paper, we will prove this empirical formula in a rigorous statistical framework, and apply it to significance tests of central England temperatures.**

***Index Terms*—Fourier power spectrum, significance tests, red noise, central England temperatures**

I. Introduction

In climatic signal processing, significance tests are used to extract statistically significant feature from randomness. To determine significance levels for Fourier spectra, one first needs to choose an appropriate background spectrum. For many geophysical phenomena, an appropriate background noise is red noise [2]. A simple model for red noise is the univariate lag-1 autoregressive AR1 process [2-7].

$$x_0 = 0,\ x_n = \lambda x_{n-1} + z_n,\ n = 1, 2, 3, ..., \qquad (1)$$

where λ is a constant, $|\lambda| < 1$ and $\{z_n\}$ is the Gaussian white noise with variance σ^2.

Let us consider the discrete Fourier transform (DFT) of the sequence $\{x_k\}_0^{N-1}$,

$$\widehat{x}_k = \frac{1}{N}\sum_{l=0}^{N-1} x_l \epsilon_k^l \quad (k = 0, 1, ...N-1) \qquad (2)$$

where $\epsilon_k = e^{-\frac{2\pi k}{N} i}$ $(k = 0, 1, ...N-1)$ and $|\widehat{x}_k|^2$ is called Fourier power spectrum. In 1998, Torrence and Compo [1] obtained the following empirical formula

$$\frac{N|\widehat{x}_k|^2}{\tilde{\sigma}^2} \quad \text{is distributed as} \quad \frac{1-\lambda^2}{2(1-2\lambda\cos\frac{2\pi k}{N}+\lambda^2)}\chi_2^2 \qquad (3)$$

where $\tilde{\sigma}^2$ is the variance of the sequence $\{x_k\}_0^{N-1}$ and χ_2^2 are chi-square distributed with two degrees of freedom.

This research was partially supported by 973 project No.2010CB950504, 863 project No. 2010AA012305, NSFC No.41076125, Fundamental Research Funds for the Central Universities (Key Program), State Key Laboratory of Earth Surface Processes and Resource Ecology, and Polar Climate and Environment Key Laboratory.

In this paper, we will give the rigorous proof of the above distribution in a rigorous statistical framework and apply it to significance tests of Central England Temperature.

II. The distribution for the Fourier power spectrum

From (1), we have that in general

$$x_k = \sum_{l=1}^{k} \lambda^{k-l} z_l \quad (k = 1, 2, 3, ...)$$

Consider the first N terms of this recurrence formula:

For convinience, we let $z_0 = 0$. Define the column vectors $\mathbb{X}$ and $\mathbb{Z}$ as

$$\mathbb{X} = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{pmatrix} \qquad \mathbb{Z} = \begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_{N-1} \end{pmatrix}$$

Let A be a square matrix of order N and

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda^2 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \lambda^{N-1} & \lambda^{N-2} & \lambda^{N-3} & \cdots & 1 \end{pmatrix} \qquad (4)$$

So

$$\mathbb{X} = A\mathbb{Z} \qquad (5)$$

Denote the discrete Fourier transform (DFT) of the sequence $\{x_k\}_0^{N-1}$ by $\{\widehat{x}_k\}_0^{N-1}$. Again let the vector

$$\widehat{\mathbb{X}} = \begin{pmatrix} \widehat{x}_0 \\ \widehat{x}_1 \\ \vdots \\ \widehat{x}_{N-1} \end{pmatrix}$$

and the matrix

$$B = \begin{pmatrix} \epsilon_0^0 & \epsilon_0^1 & \cdots & \epsilon_0^{N-1} \\ \epsilon_1^0 & \epsilon_1^1 & \cdots & \epsilon_1^{N-1} \\ \vdots & \vdots & \vdots & \vdots \\ \epsilon_{N-1}^0 & \epsilon_{N-1}^1 & \cdots & \epsilon_{N-1}^{N-1} \end{pmatrix}.$$

978-1-4244-6448-7/11/$26.00 ©2011 IEEE

Then by (2)

$$\widehat{\mathbb{X}} = \frac{1}{N} B\mathbb{X}$$

From this and (5), we have,

$$\widehat{\mathbb{X}} = \frac{1}{N} BA\mathbb{Z}$$

Let

$$C = \frac{1}{N} BA = (c_{k,j})_{k,\, j=0,1,\ldots N-1} \tag{6}$$

Then $\widehat{\mathbb{X}} = C\mathbb{Z}$, i.e.,

$$\widehat{x}_k = \sum_{j=0}^{N-1} c_{k,j} z_j \tag{7}$$

Clearly $c_{k,0} = 0$. Now we compute $c_{k,j}$.

Since $C = (c_{k,j})$ and $B = (\epsilon_k^l)$, by (4) and (6), we get

$$Nc_{k,j} = \sum_{l=j}^{N-1} \epsilon_k^l \lambda^{l-j} = \epsilon_k^j \sum_{l=j}^{N-1} (\epsilon_k \lambda)^{l-j} = \epsilon_k^j \frac{1-(\epsilon_k\lambda)^{N-j}}{1-\epsilon_k\lambda}$$

So

$$Nc_{k,j} = \frac{\epsilon_k^j - \lambda^{N-j}}{1-\epsilon_k\lambda}, \quad j = 1, 2, ..., N-1$$

By (7), we know that

$$\widehat{x}_k = \sum_{j=0}^{N-1} c_{k,j} z_j = \frac{1}{N} \sum_{j=0}^{N-1} \frac{\epsilon_k^j - \lambda^{N-j}}{1-\epsilon_k\lambda} z_j$$

So we have

$$N(1-\epsilon_k\lambda)\widehat{x}_k = \sum_{j=0}^{N-1} (\epsilon_k^j - \lambda^{N-j}) z_j \tag{8}$$

Let

$$J_k = \sqrt{N}(1-\epsilon_k\lambda)\widehat{x}_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} (\epsilon_k^j - \lambda^{N-j}) z_j$$

Taking the real part $\zeta_{1,k} = \mathrm{Re} J_k$ and the imaginary part $\zeta_{2,k} = \mathrm{Im} J_k$ of J_k, noticing that $z_0 = 0$ and each z_j is a real random variable, we have

$$\sqrt{N}(1-\epsilon_k\lambda)\widehat{x}_k = \zeta_{1,k} + i\zeta_{2,k} \tag{9}$$

Further

$$\zeta_{1,k} = \frac{1}{\sqrt{N}} \sum_{j=1}^{N-1} z_j \mathrm{Re}\{\epsilon_k^j - \lambda^{N-j}\} \tag{10}$$

Since $E(z_j) = 0$,

$$E(\zeta_{1,k}) = \frac{1}{\sqrt{N}} \sum_{j=1}^{N-1} \mathrm{Re}\{\epsilon_k^j - \lambda^{N-j}\} E(z_j) = 0 \tag{11}$$

By (10), we have

$$\zeta_{1,k} = \frac{1}{\sqrt{N}} \sum_{j=1}^{N-1} (\cos(\frac{2\pi kj}{N}) - \lambda^{N-j}) z_j$$

Again since $\{z_j\}$ is the Gaussian white noise with variance σ^2, by (11), we get,

$$\mathrm{var}(\zeta_{1,k}) = E(\zeta_{1,k}^2) = \frac{1}{N} \sum_{j=1}^{N-1} (\cos(\frac{2\pi kj}{N}) - \lambda^{N-j})^2 \sigma^2.$$

Furthermore

$$\mathrm{var}(\zeta_{1,k}) = \frac{1}{N} \sum_{j=1}^{N-1} \cos^2(\frac{2\pi kj}{N})\sigma^2 - \frac{2}{N} \sum_{j=1}^{N-1} \cos(\frac{2\pi kj}{N})\lambda^{N-j}\sigma^2$$

$$+ \frac{1}{N} \sum_{j=1}^{N-1} \lambda^{2N-2j}\sigma^2 =: I_{11} + I_{12} + I_{13} \tag{12}$$

For I_{11}, we have

$$I_{11} = \frac{1}{N} \sum_{j=1}^{N-1} \cos^2(\frac{2\pi kj}{N})\sigma^2 = \frac{1}{N} \sum_{j=1}^{N-1} \frac{\cos(\frac{4\pi kj}{N}) + 1}{2}\sigma^2$$

$$= \frac{1}{N}(\frac{N-1}{2} - \frac{1}{2})\sigma^2 = \frac{N-2}{2N}\sigma^2$$

For I_{12}, we have

$$I_{12} = -\frac{2}{N} \sum_{j=1}^{N-1} \cos(\frac{2\pi kj}{N})\lambda^{N-j}\sigma^2 = -\frac{2}{N} \sum_{j=1}^{N-1} \mathrm{Re}\{\epsilon_k^j \lambda^{N-j}\}\sigma^2$$

$$= -\frac{2}{N} \sum_{j=1}^{N-1} \mathrm{Re}\{\frac{\lambda^{N-j}}{\epsilon_k^{N-j}}\}\sigma^2 = -\frac{2}{N}\mathrm{Re}\{\frac{1-\frac{\lambda^N}{\epsilon_k^N}}{1-\frac{\lambda}{\epsilon_k}} - 1\}\sigma^2$$

So

$$I_{12} = \left(-\frac{2}{N}\mathrm{Re}\{\frac{1-\lambda^N}{1-\frac{\lambda}{\epsilon_k}}\} + \frac{2}{N}\right)\sigma^2$$

For I_{13}, we have

$$I_{13} = \frac{1}{N} \sum_{j=1}^{N-1} \lambda^{2N-2j}\sigma^2 = \frac{1}{N}\left(\frac{1-\lambda^{2N}}{1-\lambda^2} - 1\right)\sigma^2$$

Since in application, N is large enough, by $|\lambda| < 1$, we have

$$I_{11} = \frac{1}{2}\sigma^2, \quad I_{12} = 0, \quad I_{13} = 0$$

By (12)

$$\mathrm{var}(\zeta_{1,k}) = \frac{1}{2}\sigma^2 \tag{13}$$

Similarly to $\zeta_{1,k}$, by (9), we have $E(\zeta_{2,k}) = 0$ and

$$\mathrm{var}(\zeta_{2,k}) = E(\zeta_{2,k}^2) = \frac{1}{N} \sum_{j=1}^{N-1} (\sin(\frac{2\pi kj}{N}))^2\sigma^2 = \frac{1}{2}\sigma^2 \tag{14}$$

Now we compute the correlation of $\zeta_{1,k}$ and $\zeta_{2,k}$

$$E(\zeta_{1,k}\zeta_{2,k}) = -\frac{1}{N} \sum_{j=1}^{N-1} (\cos(\frac{2\pi kj}{N}) - \lambda^{N-j}) \sin(\frac{2\pi kj}{N})\sigma^2$$

Furthermore

$$E(\zeta_{1,k}\zeta_{2,k}) = -\frac{1}{N} \sum_{j=1}^{N-1} \cos(\frac{2\pi kj}{N}) \sin(\frac{2\pi kj}{N})\sigma^2$$

$$+\frac{1}{N}\sum_{j=1}^{N-1}\lambda^{N-j}\sin(\frac{2\pi kj}{N})\sigma^2 =: I_{21}+I_{22}$$

For I_{21},

$$I_{21} = -\frac{1}{N}\sum_{j=1}^{N-1}\cos(\frac{2\pi kj}{N})\sin(\frac{2\pi kj}{N})\sigma^2$$
$$= -\frac{1}{2N}\sum_{j=1}^{N-1}\sin(\frac{4\pi kj}{N})\sigma^2 = 0$$

For I_{22},

$$I_{22} = \frac{1}{N}\sum_{j=1}^{N-1}\lambda^{N-j}\sin(\frac{2\pi kj}{N})\sigma^2 = -\frac{1}{N}\sum_{j=1}^{N-1}\text{Im}\{\epsilon_k^j\lambda^{N-j}\}\sigma^2$$
$$= -\frac{1}{N}\sum_{j=1}^{N-1}\text{Im}\{\frac{\lambda^{N-j}}{\epsilon_k^{N-j}}\}\sigma^2 = -\frac{1}{N}\text{Im}\{\frac{1-\frac{\lambda^N}{\epsilon_k^N}}{1-\frac{\lambda}{\epsilon_k}}-1\}\sigma^2$$

Furthermore

$$I_{22} = -\frac{1}{N}\sigma^2\text{Im}\{\frac{1-\lambda^N}{1-\frac{\lambda}{\epsilon_k}}\}$$

Since N is large enough in application and $|\lambda|<1$, we have

$$E(\zeta_{1,k}\zeta_{2,k}) = I_{21}+I_{22} = 0 \quad (15)$$

Finally, by (9) and (13)-(15), we have

$$N|1-\epsilon_k\lambda|^2|\widehat{x}_k|^2 = \zeta_{1,k}^2+\zeta_{2,k}^2$$

where $\zeta_{1,k}$ and $\zeta_{2,k}$ are independent Gaussian random variables with mean 0 and variance $\frac{1}{2}\sigma^2$. From this, we deduce that

$$N|1-\epsilon_k\lambda|^2|\widehat{x}_k|^2 \quad \text{is distributed as} \quad \frac{1}{2}\sigma^2\chi_2^2$$

Since $|1-\epsilon_k\lambda|^2 = |1-\lambda e^{-\frac{2\pi k}{N}i}|^2 = 1-2\lambda\cos\frac{2\pi k}{N}+\lambda^2$, we get

$$\frac{N|\widehat{x}_k|^2}{\sigma^2} \quad \text{is distributed as} \quad \frac{1}{2(1-2\lambda\cos\frac{2\pi k}{N}+\lambda^2)}\chi_2^2$$

From (16) and a known result [8-9] that the relation between noise variance σ^2 and the variance $\tilde{\sigma}^2$ of the sequence $\{x_k\}_0^{N-1}$ is

$$\sigma^2 = \frac{\tilde{\sigma}^2}{1-\lambda^2},$$

we get the formula (3) immediately.

III. Application

We examine 1659-2000 Central England Temperature (CET)[10]. As suggested in [4-5], we will transform the original signal such that the pdf of the transformed data is Normal. A practical way of doing this is by taking the inverse normal cumulative distribution function (cdf). The main reason for doing it is because otherwise the normally distributed red noise null hypothesis we use is wrong. The Normalized Central England Temperature Index is shown in Figure 1. Now we will apply Formula (3) to do a significance test. We find significant peaks at about 3 year and 24 year periods well above the background spectrum (see Figure 2). This is a instinct frequency feature of Central England Temperature and can not be interpreted by climate background noise.

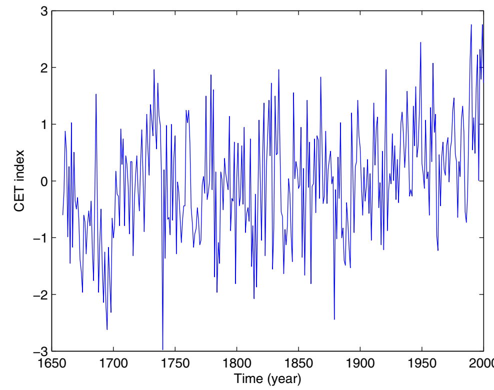

Fig. 1. Normalized Central England Temperature (CET) Index

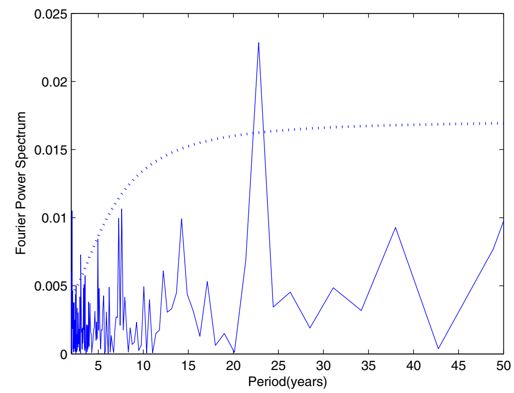

Fig. 2. Fourier power spectrum of CET index is computed by discrete Fourier transform. The dashed line is 95% confidence red noise spectrum.

IV. Conclusion

Statistical significance tests for Fourier power spectra are widely used to extract statistically significant feature from climate signals. For a long time, all statistical significance tests are based on Torrence and Compo's famous empirical formula on the distribution of Fourier power spectrum of red noise. In our study, we prove that this empirical formula is correct in a rigorous statistical framework. As application, we examine Central England Temperature and discover significant peaks at about 3 year and 24 year periods which are instinct frequency feature.

References

[1] C. Torrence and G. P. Compo, "A Practical Guide to Wavelet Analysis", *Bull. Amer. Meteor. Soc.*, vol. 79, pp. 61-78, 1998.

[2] M. E. Mann and J. M. Lees, "Robust estimation of background noise and signal detection in climatic time series," *Climate Change*, vol. 33, pp. 409-445, 1996.

[3] P. J. Brockwell and R. A. Davis, *Time series: Theory and Methods*, New York: Springer-Verlag, 1991.

[4] A. Grinsted, J. Moore, and S. Jevrejeva, "Application of the cross wavelet transform and wavelet coherence to geophysical time series," *Nonli. Processes Geophys.* vol. 11, pp. 561-566, 2003.

[5] S. Jevrejeva, J. Moore, and A. Grinsted, “Influence of the Arctic Oscillation and El Nieno-Southern Oscillation (ENSO) on ice conditions in the Baltic Sea: The wavelet approach,” *J. Geophys. Res.*, vol. 108, pp. 4677-4687, 2003.

[6] C. Chatfield, *The Analysis of Time Series: An Introduction*. New York: Chapman and Hall, 1989.

[7] D. W. J. Thompson and J. M. Wallace, “The Arctic Oscillation signature in the winter geopotential height and temperature fields,” *Geophys. Res. Lett.*, vol. 25, pp. 1297-1300, 1998.

[8] C. Franzke, “Multi-scale analysis of teleconnection indices: climate noise andnonlinear trend analysis,” *Nonlin. Processes Geophys.*, vol. 16, pp. 65-76, 2009

[9] H. Storch and F. W. Zwiers, *Statistical Analysis in Climate Research*, New York: Cambridge University Press, 1999.

[10] G. Plaut, M. Ghil, and R. Vautard, “Interannual and interdecadal variability in 335 years of central England temperatures,” *Science*, vol. 268, pp. 710-713, 1995.